Parity Violation in the Three Dimensional Thirring Model

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Abstract

We discuss parity violation in the 3-dimensional (N flavour) Thirring model. We find that the ground state fermion current in a background gauge field does not posses a well defined parity transformation. We also investigate the connection between parity violation and fermion mass generation, proving that radiative corrections force the fermions to be massive.

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Planar theories in three dimensional space-time [1] show a variety of interesting phenomena relevant not only for Quantum Field Theory but also for Condensed Matter physics.

A first important feature of three dimensional kinematics concern the possibility of giving a (topological) mass to the vector field by including an unconventional term in gauge field Lagrangians [2]-[4]. This term, of topological origin, is the Chern-Simons (CS) secondary characteristic.

Now, a salient property of the CS term is that it violates P and T invariance. Since the same happens with the mass term for a (two-component) Dirac spinor in three dimensions, it is natural to expect interesting connections between both masses in three dimensional gauge theories coupled to fermions. Indeed, in refs.[2]-[3] it was shown that if any of the two mass terms is inserted in the Lagrangian, the other is then induced by radiative corrections.

Later on, it was also shown that even massless fermions, when coupled to gauge fields, generate a CS term [5]-[6]. Originally, this effect was thought as a consequence of the introduction of a fermion mass term whithin the (Pauli-Villars) regularization procedure [6]. However, the occurrence of the CS term was confirmed in ref.[7] using the ζ-function approach, where no regulating mass-term is added at any stage of the calculations. In fact, the violation of parity in odd dimensions (and, consequently, the generation of a CS term) is analogous to the non-conservation of the axial current in even dimensions: the imposition of gauge invariance produces in both cases an anomaly for a symmetry (parity, chiral symmetry) of the original action for *any* sensible regularization.

As remarked above, the gauge field and fermion mass terms belong together since both violate P and T; in perturbation theory, one can be generated from the other. Indeed, by including a bare CS term in the QED_3 Lagrangian, Deser, Jackiw and Templeton [3] showed that even if the bare fermion mass is set to zero, the physical (renormalized) fermion mass still does not vanish (see also [8]).

In connection with these interesting phenomena, gauge models containing CS terms have attracted special attention in different areas. The possibility of giving a concrete realization of exotic spin and statistics popularized CS models in condensed matter problems like the Hall effect and high-Tc superconductivity [9]. Connections between pure CS theory with rational conformal field theories in two dimensions as well as with knot theory were also discovered [10] and considerable developements have been achieved in this area [11].

We discuss in this letter a three dimensional purely fermionic model, the (N flavour) Thirring model which, as we shall see, shares many of the attractive features described above. In fact, by introducing a vector auxiliary field to eliminate the quartic interaction, the appearence of a CS term is made apparent together with the breaking of parity and the consequent fermion mass generation.

Four fermion interaction models in 3 dimensions with N flavours are known to be renormalizable in the $1/N$ expansion [12]. Precisely, renormalizability and dynamical mass generation have been investigated in the Thirring model within the $1/N$ approximation, by introducing scalar and vector fields to split the current-current interaction [13]-[16]. Experience with 2-dimensional Thirring and Chiral Gross-Neveu models has shown the convenience of using vector fields (instead of scalar fields, as originally done in [17]) to achieve this splitting when exploring properties where symmetries play a crucial rôle [18]-[19]. In this way, the decoupling of massless excitation as well as the 2-dimensional "almost long-range order", predicted by means of the $1/N$ expansion [20] can be clearly seen without any kind of approximation. Following this route we shall show that parity violation, through the appearence of a CS terms for the auxiliary field, takes place in the 3-dimensional Thirring model. From this, generation of a fermion mass can be inferred and in fact we shall explicitly see that parity violation requires the fermion to be massive.

We start from the 3-dimensional (Euclidean) Thirring model Lagrangian:

$$\mathcal{L}_{Th} = \bar{\psi}^i i\partial\!\!\!/ \psi^i - \frac{g^2}{2N} J^\mu J_\mu \tag{1}$$

where ψ^i are N two-component Dirac spinors and J^μ the $U(1)$ current,

$$J^\mu = \bar{\psi}^i \gamma^\mu \psi^i. \tag{2}$$

The coupling constant g^2 has dimensions of inverse mass.

The partition function for the theory is defined as

$$\mathcal{Z}[B_\mu] = \mathcal{N}_1 \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[-\int (\bar{\psi}^i i\partial\!\!\!/ \psi^i - \frac{g^2}{2N} J^\mu J_\mu + J^\mu B_\mu) d^3x] \tag{3}$$

where B_μ is an external source and $\mathcal{N}_1$ is a normalization constant.

In order to evaluate $\mathcal{Z}[B_\mu]$ we first eliminate the quartic interaction by introducing a vector field A_μ through the identity

$$\exp(\int \frac{g^2}{2N} J^\mu J_\mu d^3x) = \mathcal{N}_2 \int \mathcal{D}A_\mu \exp[-\int (\frac{1}{2} A^\mu A_\mu + \frac{g}{\sqrt{N}} J^\mu A_\mu) d^3x] \quad (4)$$

(here $\mathcal{N}_2$ is some normalization constant) so that the partition function becomes

$$\mathcal{Z}[B_\mu] = \mathcal{N} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp[-\int (\bar{\psi}^i (i\partial\!\!\!/ + B\!\!\!/ + \frac{g}{\sqrt{N}} A\!\!\!/)\psi^i + \frac{1}{2} A^\mu A_\mu) d^3x]. \quad (5)$$

Before proceeding to integrate out the fermions we shift the vector field in the form

$$\frac{g}{\sqrt{N}} A_\mu + B_\mu \to \frac{g}{\sqrt{N}} A_\mu \quad (6)$$

so that we have

$$\begin{aligned} \mathcal{Z}[B_\mu] &= \mathcal{N} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp[-\int \bar{\psi}^i (i\partial\!\!\!/ + \frac{g}{\sqrt{N}} A\!\!\!/)\psi^i d^3x] \times \\ & \exp[-\frac{1}{2} \int (A_\mu - \frac{\sqrt{N}}{g} B_\mu)^2 d^3x]. \end{aligned} \quad (7)$$

The fermionic path-integral gives, as usual, a determinant

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(-\int \bar{\psi}^i (i\partial\!\!\!/ + \frac{g}{\sqrt{N}} A\!\!\!/)\psi^i d^3x) = det^N (i\partial\!\!\!/ + \frac{g}{\sqrt{N}} A\!\!\!/) \quad (8)$$

which, as it is by now well known, has a parity violating contribution in the form of a CS term. Indeed, it has been proven [5]-[7] that the fermionic determinant takes the form

$$\log det(i\partial\!\!\!/ + \frac{g}{\sqrt{N}} A\!\!\!/) = \pm \frac{ig^2}{16\pi N} \int \epsilon_{\mu\nu\alpha} F^{\mu\nu} A^\alpha d^3x + I_{PC}[A_\mu] \quad (9)$$

where I_{PC} stands for parity conserving terms.

Some comments are here in order:

1. The result (9) for massless fermions, originally obtained using Pauli-Villars regularization [6], has been reproduced by a variety of regularizations. In particular, in ref. [7] we have shown the appearence of

the CS term by using the ζ-function, a regularization scheme which makes unnecessary the introduction of a regulating mass for fermions. This approach then shows that the emergence of a (parity violating) CS term is not just a byproduct of a regularization scheme which (as the Pauli-Villars one), breaks explicitly parity (through the introduction of regulating fermion mass terms) but signals the impossibility of mantaining in three dimensional gauge theories both gauge and parity invariance.

2. More important for our purpose is to discuss the origin of double sign in (9). Within the Pauli-Villars scheme it arises due to an ambiguity when taking the limit of the regulator fermion mass going to zero. Concerning the ζ-function approach, the sign ambiguity can be traced back to the choice of an integration path Γ in the complex plane, necessary for defining the complex powers of the Dirac operator [21]. In odd dimensions, the choice of Γ in the upper (lower) half plane results in a positive (negative) overall sign (see ref. [7] for details).

   Again, the sign ambiguity is not a byproduct of a particular regularization scheme but an intrinsic feature of regularization in odd dimensional spaces.

   We then consider that the only consistent way of taking into account the sign ambiguity of the CS term arising from each one of the N fermion determinants is to assign *the same sign* to all of them. Any other alternative, as for example to take (for even N) one half of "+" signs and one half of "$-$" signs (so that the overall CS term would be absent, as advocated in [15]) would mean that one has to define complex powers of the Dirac operator in different ways for different fermion species, this being in our opinion not mathematically sound.

3. The result (9) was obtained in [6] by computing the fermion current at one (fermion) loop; it is argued that no further radiative corrections arise from higher order loops. Calculations were performed in that work for a constant and static $F_{\mu\nu}$. In [7] the CS term was obtained in a *non-perturbative* way and for *arbitrary* $F_{\mu\nu}$.

We shall not consider in what follows the (parity conserving) terms included in $I_{PC}[A_\mu]$ since they will play no rôle in the analysis of parity violation. Up to one loop one has for example [6],

$$I_{PC}[A_\mu] = \frac{\zeta(3/2)}{\pi^2} \int (\frac{g}{2\sqrt{N}} F^2_{\mu\nu})^{3/2} d^3x. \tag{10}$$

We use now the result (9) to write the partition function (7) in the form

$$\mathcal{Z}[B_\mu] = \mathcal{N} \exp(-\frac{1}{2}\frac{N}{g^2} \int B_\mu B^\mu d^3x) \int \mathcal{D}A_\mu \exp(-S_{eff}[A_\mu]) \tag{11}$$

where $S_{eff}[A_\mu]$ is given by

$$S_{eff} = \frac{1}{2} \int A_\mu S^{\mu\nu} A_\nu d^3x - \frac{\sqrt{N}}{g} \int A_\mu B^\mu d^3x \tag{12}$$

and

$$S^{\mu\nu} = \delta^{\mu\nu} \mp \frac{ig^2}{4\pi} \epsilon^{\mu\alpha\nu} \partial_\alpha. \tag{13}$$

Performing the gaussian integral one readily obtains

$$\mathcal{Z}[B_\mu] = \mathcal{N} \exp(-\frac{1}{2}\frac{N}{g^2} \int B_\mu B^\mu d^3x) \exp(\frac{1}{2}\frac{N}{g^2} \int B_\mu G^{\mu\nu} B_\nu d^3x d^3y) \tag{14}$$

where $G^{\mu\nu}$ is the Green function for $S_{\mu\nu}$, whose Fourier transform is

$$\bar{G}^{\nu\alpha}(k) = \frac{1}{1+a^2k^2} (\delta^{\nu\alpha} + a^2 k^\nu k^\alpha \mp a \epsilon^{\nu\rho\alpha} k_\rho). \tag{15}$$

Here $a = g^2/4\pi$. In coordinate space $G^{\nu\alpha}$ reads

$$G^{\nu\alpha}(x) = \frac{4\pi^3}{g^4} \frac{\exp(-\frac{|x|}{a})}{|x|} \delta^{\nu\alpha} - \frac{1}{4\pi} \partial^\nu \partial^\alpha (\frac{\exp(-\frac{|x|}{a})}{|x|}) \pm \frac{i\pi}{g^2} \epsilon^{\nu\rho\alpha} \partial_\rho (\frac{\exp(-\frac{|x|}{a})}{|x|}). \tag{16}$$

Let us first compute the fermion current expectation value in the B_μ background:

$$\langle J_\mu(x) \rangle_B = -\frac{1}{\mathcal{Z}} \frac{\delta \mathcal{Z}}{\delta B^\mu} = \frac{N}{g^2} B_\mu(x) - \frac{N}{g^2} \int G_{\mu\alpha}(x-y) B^\alpha(y) d^3y. \tag{17}$$

The parity violating term in J_μ takes then the form

$$\langle J^{PV}_\mu \rangle = \pm \frac{N}{g^4} \int d^3y \frac{\exp(-|x-y|/a)}{|x-y|} {}^* F_\mu(y) \tag{18}$$

where

$$ {}^*F_\mu(x) = i\epsilon_{\mu\rho\alpha}\partial^\rho B^\alpha(x). \tag{19}$$

We can readily check that in the $g^2 \to 0$ limit we recover the well known result for non self-interacting fermions [5]-[7]. To this end we note that the (three dimensional) Dirac δ-function can be represented as

$$\lim_{a\to 0} \frac{1}{4\pi a^2}\frac{\exp(-|x-y|/|a|)}{|x-y|} = \delta^{(3)}(x-y), \tag{20}$$

so that one gets in the $g^2 \to 0$ limit:

$$\langle J_\mu^{PV}\rangle_{g^2\to 0} = \pm\frac{N}{4\pi}{}^*F_\mu \tag{21}$$

in complete agreement with the non self-interacting result [5]-[7].

The Thirring model Lagrangian $\mathcal{L}_{Th}$ defined in eq.(1) is invariant under parity. We have found however that at the quantum level a physical quantity, the ground state current in a gauge-field background $\langle J_\mu\rangle_B$, does not posses a well defined parity transformation since there is a contribution $\langle J_\mu^{PV}\rangle_B$, given by (18), which transforms as a pseudovector. Hence, parity is spontaneously broken. One should note that although anomalous, $\langle J_\mu^{PV}\rangle$ is conserved, as one easily checks from eqs.(18)-(19).

Our result (18) is nothing but the extension to the case where there is a fermion self-interaction of those in refs. [2]-[6], where parity violation through the generation of a CS term was discovered for QED_3 and QCD_3. Let us insist that we have adopted the point of view that the N fermion contributions have to be considered with the same sign at the light of the ζ-function regularization analysis.

Let us now show that the resulting parity violation renders the fermion massive. To this end, instead of the partition function (5) we consider

$$\mathcal{Z}[\bar\eta,\eta] = \mathcal{N}\int \mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}A_\mu \exp[-\int(\bar\psi^i(i\partial\!\!\!/ + \frac{g}{\sqrt{N}}A\!\!\!/)\psi^i + \bar\eta\psi + \bar\psi\eta + \frac{1}{2}A^\mu A_\mu)d^3x] \tag{22}$$

where $\bar\eta$ and η are anticommuting sources. As before, we integrate out the fermions and solely consider the parity violating contribution to the fermion determinant. We thus have

$$\mathcal{Z}[\bar\eta,\eta] = \int \mathcal{D}A_\mu \quad \exp[\int(-\frac{1}{2}A_\mu A^\mu \pm \frac{ig^2}{16\pi^2}\int \epsilon_{\mu\nu\alpha}F^{\mu\nu}A^\alpha)d^3x] \times$$

$$\exp[\int \bar{\eta}(x) G_F(x-y)\eta(y) d^3x d^3y] \tag{23}$$

Here, $G_F(x-y)$ is the Dirac operator Green function,

$$G_F(x-y) = \frac{1}{i\partial\!\!\!/ + \frac{g}{\sqrt{N}} A\!\!\!/}\delta^3(x-y). \tag{24}$$

We are now ready to evaluate the fermion propagator,

$$S_F(x-y) = \langle \psi(x)\bar{\psi}(y)\rangle = \frac{1}{\mathcal{Z}}\frac{\delta^2 \mathcal{Z}}{\delta\bar{\eta}(x)\eta(y)}\bigg|_{\bar{\eta}=\eta=0}, \tag{25}$$

which can be written in the form

$$S_F(x-y) = \int \mathcal{D}A_\mu G_F(x-y)\exp[\int(-\frac{1}{2}A_\mu A^\mu \pm \frac{ig^2}{16\pi}\epsilon_{\mu\nu\alpha}F^{\mu\nu}A^\alpha)d^3x] \tag{26}$$

We now expand G_F in powers of $g/\sqrt{N}$, integrate over A_μ and Fourier transform, getting

$$S_F(p) = \frac{-1}{p\!\!\!/} + \frac{1}{N}\frac{-1}{p\!\!\!/}\Sigma(p)\frac{-1}{p\!\!\!/} + O(1/N^2) \tag{27}$$

with the fermion self-energy given by

$$\Sigma(p) = g^2 \int \frac{d^3k}{(2\pi)^3}\gamma_\mu \frac{1}{k\!\!\!/ - p\!\!\!/}\gamma_\nu \bar{G}^{\mu\nu}(k). \tag{28}$$

Note that the A_μ propagator $\bar{G}^{\mu\nu}(k)$ (given by eq.(15)) depends on g^2 and that our approximation corresponds effectively to a $1/N$ expansion.

Now, from renormalization theory, one knows that the fermion physical mass is determined by the equation [22]

$$m = -\frac{1}{N}\Sigma(p)|_{p\!\!\!/=m}. \tag{29}$$

To order $1/N$, we can set $m=0$ in the r.h.s. of eq.(29) so that we get

$$m = -\frac{1}{N}\Sigma(p)|_{p\!\!\!/=0} = \mp\frac{1}{N\pi^2}\int d^3k \frac{1}{k^2 + (\frac{4\pi}{g^2})^2} \tag{30}$$

a result which picks as sole contribution that coming from the parity violating term in $\bar{G}_{\mu\nu}$. As expected, there is a linear divergence which will be handled by introducing a Pauli-Villars massive vector field (with mass M). Then, the regularized form of eq.(30) takes the form

$$m = \mp \frac{2}{N}|M| + O(1/M). \tag{31}$$

It has been established, within the $1/N$ approximation [12]-[14], that the 3-dimensional Thirring model is renormalizable. We can then make sense from the divergent result (31) adopting the following viewpoint [22]: as parity is violated there is no reason for the fermion to remain massless since it eventually acquires a (divergent) mass (given by eq.(31)). One should then start from a Lagrangian with a (parity violating) fermion bare mass term and see whether radiative corrections yield a finite renormalized mass.

Now, if one starts with fermions having a bare mass m_0, then eq.(9) should be replaced by [5, 6]

$$\log det(i\partial\!\!\!/ + m_0 + \frac{g}{\sqrt{N}}\, A\!\!\!/) = \frac{m_0}{|m_0|}\frac{ig^2}{16\pi N}\int \epsilon_{\mu\nu\alpha}F^{\mu\nu}A^\alpha d^3x + I_{PC}[A_\mu, m_0] \tag{32}$$

Notice that the double sign arising in the massless case has been traded by the sign of the bare mass.

One can now repeat all the steps leading to eq.(27). For the renormalized mass m, instead of eq.(29), one has [22]

$$m = m_0 - \frac{1}{N}\Sigma'(p)|_{p\!\!\!/=m} \tag{33}$$

where $\Sigma'(p)$ is now the self-energy corresponding to massive fermions. Taking for simplicity m_0 to be of order $1/N$, one can simply write

$$m = m_0 - \frac{1}{N}\Sigma(p)|_{p\!\!\!/=0}. \tag{34}$$

From this we get, after Pauli-Villar regularization of $\Sigma(p)$,

$$m = m_0 - \frac{m_0}{|m_0|}\frac{2}{N}|M| + O(1/M). \tag{35}$$

This result is highly non trivial in the sense that the opposite signs of the two terms in (35) make it possible to define a finite value for the fermion mass.

Had both contributions had the same sign, one would have never attained a finite value for the fermion mass through radiative corrections. We consider this result as a proof of the fact that, due to parity violation, fermions do acquire a mass in the 3-dimensional Thirring model.

In summary, we have analysed parity violation in the 3-dimensional (N flavour) Thirring model. We have found that the ground state current in a gauge field background does not posses a well defined parity transformation. We have also studied the connection between parity violation and mass generation for fermions. By evaluating the fermion self-energy we have shown that radiative corrections force the fermion to be massive.

Our approach is based on the introduction of an auxiliary vector field followed by integration of fermions. This makes apparent in a very simple way parity violation through the occurrence of a CS term. It is important to notice that the route we followed takes into account parity violating contributions *exactly*. Indeed, the ζ-function approach to the evaluation of the fermionic determinant is not just a one loop approximation but a complete result for parity violating terms [7] (of course the parity conserving terms cannot be evaluated in a closed form).

The results above should be compared with those of ref. [14]-[16]. In [14] the CS contribution was not taken into account for the case of massless fermions (this in contrast with the results of [3], [5]-[7] where it was shown that even massless fermions generate a CS term). In [15], for an even number of flavours, the total CS contribution was cancelled by use of regularization sign ambiguities in a way that we find rather artificial and that has been shown to be energetically unfavorable in ref.[16]. Finally, our conclusions about parity violation and mass generation agree with those obtained in [16], where the quartic interaction was splitted using scalar fields and the resulting effective theory was analysed within the $1/N$ expansion.

Let us end by noting that our approach can be straightforwardly extended to the case of $U(N)$ Thirring model. We hope to report on this issue in a separate work.

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